MIMO Channel Capacity in Co-Channel Interference

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Abstract— **Recent information theory results have indicated that a large channel capacity exists for wireless systems with multiple transmit and receive antennas. With different assumptions of channel knowledge and interference knowledge at the transmitter, the channel capacity of multiple input multiple output (MIMO) systems has been studied under both spatially white and colored interference and noise. In this paper, we fix the total interference-plus-noise power and evaluate the outage capacity under two spatially colored interference environments: (1) a few high-data-rate interferers each with high power, (2) a large number of low-data-rate interferers each with low power. The results show that MIMO capacity is larger with fewer high-data-rate interferers. We also assess the impact of an estimated channel and/or interference on capacity. In the case of 4 transmit and 4 receive antennas for the user of interest, 10 interferers, total-interference-to-noise ratio and signal-to-noise ratio are both 20dB, the results show that it is beneficial to estimate the channel and/or interference if the variance of estimation error is less than about 50% of the variance of true channel and/or interference.**

I. Introduction

Recent information theory results have indicated that a large channel capacity exists for wireless systems with multiple transmit and receive antennas [1]. With different assumptions of channel knowledge and interference knowledge at the transmitter, the channel capacity of multiple input multiple output (MIMO) systems have been studied under both spatially white and colored interference and noise by applying different power allocation schemes at the transmitter [2][3]. Meanwhile, in future generation wireless communication systems, multi-rate data services will be dominant. To support users of different data rate at a certain quality of service (e.g., a certain level of bit error rate), the user's transmit power is, in general, proportional to the data rate. Therefore, high-data-rate users need high transmit powers.

In multiple-access systems, the interference is, in general, spatially colored. In this paper, we fix the total interference-plus-noise power and examine the MIMO outage capacity under two spatially colored interference environments: (1) a few high-data-rate interferers each with high power, (2) a large number of low-data-rate interferers each with low power. We would like to find out under which interference environment a MIMO system achieves a higher outage capacity. The assumption of fixed total interference-plus-noise power is reasonable since in wireless systems, likely there is some mechanism, such as power control, to control the interference experienced by a user. The result of this work has the implication on design of the medium access control (MAC) protocols and scheduling of packet transmissions in future wireless systems. We will also assess the impact of an estimated channel and/or interference on capacity.

II. System Model

We consider a single-user narrowband link with interference from other users. The user of interest is equipped with M transmit and N receive antennas. Each interfering user has one transmit antenna, and the same N receive antennas as the user of interest. The received signal vector $\mathbf{y}$ $(N \times 1)$ is

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \underbrace{\sqrt{\frac{P_I}{L}}\sum_{i=1}^{L}\mathbf{h}_i s_i + \mathbf{w}}_{\mathbf{n}} \quad (1)$$

where $\mathbf{H}$ $(N \times M)$ is the MIMO channel matrix of the user of interest, $\mathbf{s}$ $(M \times 1)$ is the transmit signal vector of the user of interest, $\mathbf{n}$ $(N \times 1)$ is the interference-plus-noise vector at the receiver. The number of interferers is L, P_I is the fixed total interference power, $\mathbf{h}_i$ $(N \times 1)$ is the channel vector of the ith interferer, s_i is the ith interferer's transmit signal with unit power, and $\mathbf{w}$ $(N \times 1)$ is the thermal noise with covariance matrix $\mathbf{E}\{\mathbf{w}\mathbf{w}^\dagger\} = \sigma^2\mathbf{I}_N$ where $\dagger$ denotes transpose conjugate. The channel matrix $\mathbf{H}$ and the channel vectors $\mathbf{h}_i$'s are mutually independent, and assumed to be quasi-static (constant over one frame) having uncorrelated realizations in different frames. It is further assumed that the elements in $\mathbf{H}$ and $\mathbf{h}_i$ are identically independent distributed (i.i.d.) complex Gaussian random variables (RVs) with zero-mean and unit variance. This implies flat Rayleigh fading and that antennas are separated far apart. The signal of the user of interest $\mathbf{s}$, the interfering signal $\mathbf{s}_i$, and the thermal noise $\mathbf{w}$ are mutually independent. It is obvious in (1) that each interferer has the same power. More interferers in the system, lower power each interferer has. It can be shown that the covariance matrix of the interference-plus-noise is

$$\mathbf{R}_0 = \mathbf{E}\{\mathbf{n}\mathbf{n}^\dagger\} = \frac{P_I}{L}\sum_{i=1}^{L}\mathbf{h}_i\mathbf{h}_i^\dagger + \sigma^2\mathbf{I}_N, \quad (2)$$

and the covariance matrix of the received signal is

$$\mathbf{E}\{\mathbf{y}\mathbf{y}^\dagger\} = \mathbf{H}\Sigma_s\mathbf{H}^\dagger + \mathbf{R}_0 \quad (3)$$

where $\Sigma_s = \mathbf{E}\{\mathbf{s}\mathbf{s}^\dagger\}$.

In our system model, we assume each interferer has one transmit antenna. However, it is easy to accommodate interferers with more than one transmit antenna by aggregating several interfering users with one transmit antenna.

III. Channel Capacity

In this section, we derive the MIMO channel capacity with spatially colored interference and under different assumptions of channel and interference knowledge at the transmitter: (1) both channel and interference covariance matrices $\mathbf{H}$ and $\mathbf{R}_0$ are available, (2) only $\mathbf{H}$ is available, and (3) neither $\mathbf{H}$ nor $\mathbf{R}_0$ is available. In all the cases, we assume that the receiver knows the channel $\mathbf{H}$. Comparing to [3], our derivation uses a modeled interference covariance matrix as (2). In addition, we give a new interpretation of MIMO channel capacity under spatially colored interference.

We introduce the differential entropy of a circularly symmetric complex Gaussian random vector. If $\mathbf{x}$ is a circularly symmetric complex Gaussian random vector with covariance matrix $\mathbf{Q}$, the differential entropy of $\mathbf{x}$ is $\log_2 \det(\pi e \mathbf{Q})$. In addition, circularly symmetric complex Gaussians are entropy maximizers [4].

Assuming the interference-plus-noise $\mathbf{n}$ in (1) is circularly symmetric complex Gaussian, the optimal distribution for the signal $\mathbf{s}$ is then circularly symmetric complex Gaussian [4] [5]. As the receiver knows the channel, the mutual information between the channel input and output is given as

$$\begin{aligned} &\mathcal{I}(\mathbf{s};\mathbf{y}) \\ =&\log_2 \det\left[\pi e(\mathbf{H}\Sigma_s\mathbf{H}^\dagger + \mathbf{R}_0)\right] - \log_2 \det(\pi e \mathbf{R}_0) \\ =&\log_2 \det\left(\mathbf{I}_N + \mathbf{R}^{-1}\mathbf{H}\frac{\Sigma_s}{\sigma^2}\mathbf{H}^\dagger\right) \\ =&\log_2 \det\left[\mathbf{I}_N + \left(\mathbf{R}^{-1/2}\mathbf{H}\right)\frac{\Sigma_s}{\sigma^2}\left(\mathbf{R}^{-1/2}\mathbf{H}\right)^\dagger\right] \end{aligned} \tag{4}$$

where

$$\mathbf{R} = \frac{P_I}{\sigma^2 \cdot L}\sum_{i=1}^{L}\mathbf{h}_i\mathbf{h}_i^\dagger + \mathbf{I}_N, \tag{5}$$

and the third equality comes from the facts that $\det(\mathbf{I} + AB) = \det(\mathbf{I} + BA)$ for square matrices A and B and $\mathbf{R}^{-1/2}$ is Hermitian. We denote $\frac{P_I}{\sigma^2}$ as the ratio of total interference power to noise power. The channel capacity is the maximized mutual information with transmit power constraint $tr(\Sigma_s) \leq P_T$, i.e.,

$$C = \max_{tr(\frac{\Sigma_s}{\sigma^2}) \leq \frac{P_T}{\sigma^2}} \log_2 \det\left[\mathbf{I}_N + \left(\mathbf{R}^{-1/2}\mathbf{H}\right)\frac{\Sigma_s}{\sigma^2}\left(\mathbf{R}^{-1/2}\mathbf{H}\right)^\dagger\right], \tag{6}$$

where $\frac{P_T}{\sigma^2}$ is the ratio of signal power to noise power.

Eqn. (6) suggests that we could consider $\mathbf{R}^{-1/2}\mathbf{H}$ as a combined channel. As a result, the capacity in (6) is equivalent to the capacity of the combined channel $\mathbf{R}^{-1/2}\mathbf{H}$ under spatially white noise. With this new interpretation and the results of channel capacity under spatially white noise in [2], we obtain the capacity with spatially colored interference and noise.

A. Both channel and interference information at the transmitter

By applying water-filling power allocation with the combined channel $\mathbf{R}^{-1/2}\mathbf{H}$ at the transmitter, the channel capacity is

$$C = \sum_{i=1}^{M}\log_2(1 + p_i\lambda_i), \tag{7}$$

the optimal transmit signal covariance matrix is

$$\Sigma_s = \sigma^2\mathbf{U}\ \mathrm{diag}(p_1, \ldots, p_M)\mathbf{U}^\dagger \tag{8}$$

where

$$\mathbf{H}^\dagger\mathbf{R}^{-1}\mathbf{H} = \mathbf{U}\Lambda\mathbf{U}^\dagger, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_M), \tag{9}$$

$\lambda_1, \ldots, \lambda_M$ are the eigenvalues of $\mathbf{H}^\dagger\mathbf{R}^{-1}\mathbf{H}$, $\mathbf{U}$ is an unitary matrix consisting of eigenvectors of $\mathbf{H}^\dagger\mathbf{R}^{-1}\mathbf{H}$,

$$p_i = (\mu - \frac{1}{\lambda_i})^+ \tag{10}$$

where μ is chosen such that

$$\sum_{i=1}^{M} p_i = \frac{P_T}{\sigma^2}, \tag{11}$$

and $(x)^+$ denotes the larger of 0 and x.

B. Neither channel nor interference information at the transmitter

If the transmitter applies uniform power allocation across the transmit antennas, i.e., $\Sigma_s = (P_T/M)\mathbf{I}_M$, the capacity is given by

$$C = \log_2 \det\left(\mathbf{I}_N + \frac{P_T}{M \cdot \sigma^2}\mathbf{R}^{-1}\mathbf{H}\mathbf{H}^\dagger\right). \tag{12}$$

C. Only channel information at the transmitter

It is claimed in [3] that the optimal power allocation is the water-filling using $\mathbf{H}$ and assuming interference covariance matrix to be an identity matrix, i.e., the optimal transmit signal covariance matrix Σ_s is obtained from (8)-(11) by setting $\mathbf{R} = \mathbf{I}_N$, and the capacity is obtained by substituting the resultant Σ_s into (4). However, no justification that this scheme is optimal was given in [3]. At the same time, if we consider $\mathbf{R}^{-1/2}\mathbf{H}$ as a combined channel, without knowing $\mathbf{R}$, we do not know this combined channel. As a result, uniform power allocation at the transmitter should be used, i.e., the capacity is as (12). It is not obvious which power allocation scheme gives a higher capacity, uniform power allocation (Section III-B) or water-filling (Section III-A) using $\mathbf{R} = \mathbf{I}_N$. Uniform power allocation does not use the known channel information, while the water-filling scheme uses the incorrect interference information. We simulated 10,000 sets of $\mathbf{H}$ and $\mathbf{R}$, and in all cases water-filling scheme using $\mathbf{H}$ only gives a higher capacity than uniform power allocation. However, no proof that this is true in general has been found.

IV. Channel Capacity with Estimated Channel and Interference

When the transmitter is provided with estimates of channel and/or interference covariance matrix, we can calculate the capacity by applying water-filling as in Section III-A using estimated interference covariance and channel matrices $\hat{\mathbf{R}}$ and $\hat{\mathbf{H}}$, respectively. As a result, we are able to evaluate the degradation of capacity due to estimation error of channel and interference covariance matrices.

We model the estimate of $\mathbf{H}$ as

$$\hat{\mathbf{H}} = \mathbf{H} + \mathbf{E}_H \tag{13}$$

where $\mathbf{H}$ is the true channel. The elements in the estimation error matrix, $\mathbf{E}_H$, are i.i.d. zero-mean complex Gaussian. This implies that the estimation errors of channel are mutually independent. We assume that the variance of estimation error is proportional to the variance of true channel. Therefore, the variance of the (i, j)th element of $\mathbf{E}_H$, $\mathbf{E}_{H,ij}$, is specified by

$$\mathrm{VAR}(\mathbf{E}_{H,ij}) = \mu_H \cdot \mathrm{VAR}(\mathbf{H}_{ij}) \tag{14}$$

where μ_H is the parameter that controls the quality of the estimate. As the (i,j)th element in $\mathbf{H}$, $\mathbf{H}_{ij}$, is complex Gaussian with unit variance, $\mathrm{VAR}(\mathbf{H}_{ij}) = 1$.

Similarly, we model the estimate of $\mathbf{R}$ as

$$\hat{\mathbf{R}} = \mathbf{R} + \mathbf{E}_R \tag{15}$$

where $\mathbf{R}$ is the true interference covariance matrix. We restrict the estimation error matrix $\mathbf{E}_R$ to be Hermitian. We assume that the elements in the lower triangle of $\mathbf{E}_R$, $\mathbf{E}_{R,ij}$ for $i \leq j$, are mutually independent. The elements $\mathbf{E}_{R,ij}$ for $i < j$ are i.i.d. complex Gaussian, while the diagonal elements of $\mathbf{E}_R$ are i.i.d. real Gaussian. Again, the variance of $\mathbf{E}_{R,ij}$ is specified by

$$\mathrm{VAR}(\mathbf{E}_{R,ij}) = \mu_R \cdot \mathrm{VAR}(\mathbf{R}_{ij}). \tag{16}$$

The variance of the diagonal elements in $\mathbf{R}$ can be calculated as

$$\mathrm{VAR}(\mathbf{R}_{jj}) = \left(\frac{P_I}{\sigma^2 \cdot L}\right)^2 \sum_{i=1}^{L} \mathrm{VAR}(\mathbf{h}_{ij}\mathbf{h}_{ij}^{\dagger}) \tag{17}$$

where $\mathbf{h}_{ij}$ is the jth element in vector $\mathbf{h}_i$. Since $\mathbf{h}_{ij}$ is zero-mean complex Gaussian with unit variance, $\mathbf{h}_{ij}\mathbf{h}_{ij}^{\dagger}$ is chi-square distributed with 2 degree of freedom, and $\mathrm{VAR}(\mathbf{h}_{ij}\mathbf{h}_{ij}^{\dagger}) = 1$. As $\mathbf{h}_{ij}$'s are i.i.d. for all i and j, we have

$$\mathrm{VAR}(\mathbf{R}_{jj}) = \left(\frac{P_I}{\sigma^2}\right)^2 \frac{1}{L}. \tag{18}$$

The variance of off-diagonal elements in $\mathbf{R}$ is

$$\mathrm{VAR}(\mathbf{R}_{j_1 j_2}) = \left(\frac{P_I}{\sigma^2 \cdot L}\right)^2 \sum_{i=1}^{L} \mathrm{VAR}(\mathbf{h}_{ij_1}\mathbf{h}_{ij_2}^{\dagger}). \tag{19}$$

Let $\mathbf{h}_{ij_1} = a_1 + jb_1$, $\mathbf{h}_{ij_2} = a_2 + jb_2$, and a_1, a_2,b_1 and b_2 are i.i.d. zero-mean complex Gaussian with unit variance. It can be shown that $E(\mathbf{h}_{ij_1}\mathbf{h}_{ij_2}^{\dagger}) = 0$ and $\mathrm{VAR}(\mathbf{h}_{ij_1}\mathbf{h}_{ij_2}^{\dagger}) = 1$.

With specified μ_H and μ_R, we are able to simulate estimated channel and interference covariance matrices $\hat{\mathbf{H}}$ and $\hat{\mathbf{R}}$, respectively. The optimal transmit signal covariance matrix Σ_s is found by applying water-filling, i.e., (8)-(11) with estimates $\hat{\mathbf{H}}$ and/or $\hat{\mathbf{R}}$. The capacity is then obtained by substituting the resultant Σ_s into (4).

V. Simulation Results

We calculate the capacity under different assumptions of knowledge of channel and interference at the transmitter. For the case of only channel information at the transmitter, we use (8)-(11) and set $\mathbf{R} = \mathbf{I}_N$ to obtain the capacity. As $\mathbf{H}$ and $\mathbf{R}$ are random matrices, the capacity is treated as a random variable. The performance measurement here is the 10% outage capacity, $C_{0.1}$, where $P(C < C_{0.1}) = 10\%$. Monte Carlo simulation is used to obtain the 10% outage capacity.

In Fig. 1, we fix the total interference power and evaluate the outage capacity as the number of interferers increases. The user of interest is assumed to have 4 transmit and 4 receive antennas. The ratio of signal power to noise power and the ratio of total interference power to noise power are both 20dB. We find that the 10% outage capacity decreases significantly as the number of interferers increases. When the channel and interference are not known at the transmitter, the capacity with one interferer is 16 bps/Hz. This number is reduced sharply to 3 bps/Hz with 10 interferers each with one tenth the power. This implies that MIMO systems perform more efficiently where there are a few strong interferers.

In Fig. 2, we fix the number of interferers to be 4, and examine the outage capacity as we increase the ratio of total interference power to noise power. Again, the user of interest is assumed to have 4 transmit and 4 receive antennas. The ratio of signal power to noise power is 15dB. We observe that when the total interference power is low, knowing only the channel allows us to achieve about the same capacity as that in the case of full knowledge of channel and interference. However, when the total interference power is high, without interference information, knowing only the channel leads to about the same capacity as that in the case of no channel and interference knowledge at the transmitter.

In Fig. 3, assuming the user of interest has the same number of transmit and receive antennas, we calculate the outage capacity as the number of transmit antennas increases. We fix the number of interferers to be 4, the ratio of signal power to noise power and the ratio of total interference power to noise power are both 20dB. We observe that the capacity increases almost linearly as the number of antennas. In addition, the differences in capacity under different knowledge of channel and interference increase as the number of antennas increases.

In Fig. 4, we assess the degradation of channel capacity using estimated channel and/or interference. The user of interest is assumed to have 4 transmit and 4 receive antennas, and the ratio of signal power to noise power and the ratio of total interference power to noise power are both 20dB. The number of interferers is 10. In the case of no knowledge of channel and interference at the transmitter, we compare the capacity of uniform power allocation to that of water-filling using estimated channel and interference. We observe that, for $\mu_H = \mu_R$, when μ_H and μ_R are less than 50%, water-filling using estimated channel and interference achieves a higher capacity than uniform power allocation. In the case of only channel information at the transmitter, we compare the capacity of water-filling using known channel and estimated interference to that of water-filling using channel only. Again, for $\mu_H = \mu_R$, we observe that when μ_R is less than 50%, water-filling using known channel information and estimate of interference covariance matrix is better than water-filling using channel information only. Fig. 4 also shows the degradation of capacity due to estimation error of channel and interference for cases of $\mu_H = 0.1\mu_R$ and $\mu_H = 10\mu_R$.

VI. Conclusions

In this paper, we fixed the total interference-plus-noise power and examined MIMO outage capacity under different interference environments: (1) a few high-data-rate interferers each with high power, (2) a large number of low-data-rate interferers each with low power. The results show that MIMO capacity is larger with fewer high-data-rate interferers. We also evaluated the degradation of outage capacity using estimated channel and/or interference.

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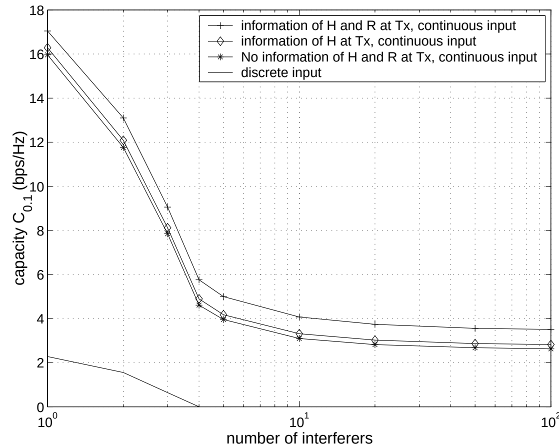

Fig. 1. 10% outage capacity versus number of interferers. The user of interest has 4 transmit and 4 receive antennas, the ratio of signal power to noise power is 20dB, and the ratio of total interference power to noise power is 20dB.

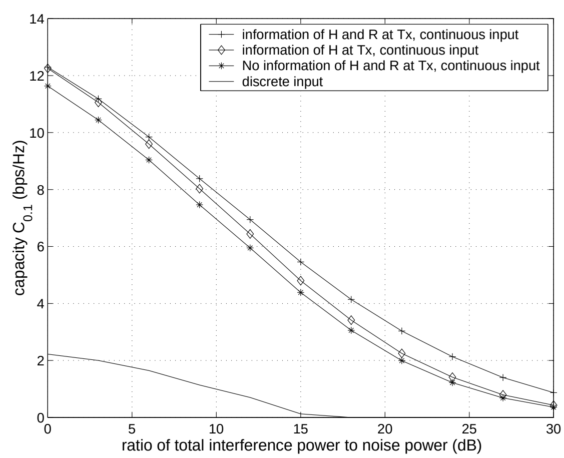

Fig. 2. 10% outage capacity versus the ratio of total interference power to noise power. The user of interest has 4 transmit and 4 receive antennas, the ratio of signal power to noise power is 15dB, and the number of interferers is 4.

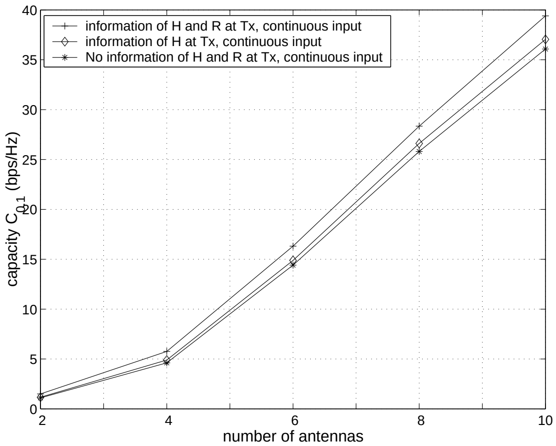

Fig. 3. 10% outage capacity versus number of antennas, assuming the user of interest has the same number of transmit and receive antennas. The number of interferers is 4, the ratio of signal power to noise power is 20dB, and the ratio of total interference power to noise power is 20dB.

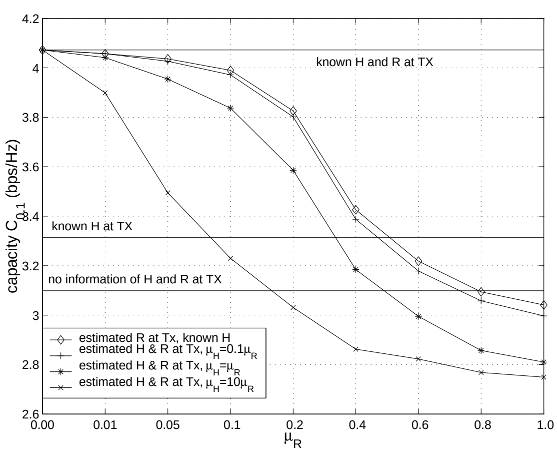

Fig. 4. 10% outage capacity versus μ_R. The user of interest has 4 transmit and 4 receive antennas, the number of interferers is 10, the ratio of signal power to noise power is 20dB, and the ratio of total interference power to noise power is 20dB.