Rectangular Information Lossless Linear Dispersion Codes

Jinsong Wu, *Member, IEEE*, and Steven D. Blostein, *Senior Member, IEEE*

***Abstract*—This paper extends square $M \times M$ linear dispersion codes (LDC) proposed by Hassibi and Hochwald to $T \times M$ non-square linear dispersion codes of the same rate M, termed uniform LDC, or U-LDC. This paper establishes a unitary property of arbitrary rectangular U-LDC encoding matrices and determines their connection to the traceless minimal non-orthogonality criterion for space-time codes. The U-LDC are then applied to rapid fading channels by constructing trace-orthonormal versions, or TON-U-LDC for $2L$ and $4L$ input symbols, where L is a positive integer. Compared to a variety of state-of-the-art codes, the proposed codes are found to perform well in both block and rapid fading channels. In rapid fading, the symbol-wise time diversity order of a $T \times M$, TON-U-LDC for $2L$ input symbols is shown to be min(T,$2M$).**

***Index Terms*—Space-time coding, MIMO systems, transmit diversity.**

I. Introduction

HASSIBI and Hochwald have proposed linear dispersion codes (LDC) as a general framework for arbitrary complex space time codes for block flat-fading channels[1]. In recent years, a number of high-rate block based complex space-time code designs have been have been proposed, including [1]–[4]. However, existing high-rate complex space-time codes have limited choices of size and have been mainly applied to block fading channels. To better exploit available time diversity, suitable designs for rapid (i.e., fast) fading channels are also needed. In block fading channels, this may be achieved by interleaving input symbols within one space-time codeword over multiple fading blocks. Even with the use of symbol-interleaving, however, it is not guaranteed that a codeword would necessarily benefit from the available diversity due to rapid fading since wireless channels are highly dynamic. The design of space-time codes that operate well in both block fading and rapid fading channels is therefore critical. In this paper, an approach to better exploit time diversity in rapid fading is investigated, one that employs more flexibly-sized rectangular algebraic code designs.

Hassibi and Hochwald proposed a rate-M linear dispersion codes (LDC) of arbitrary square size $M \times M$ in [1], Eq. (31). This paper extends [1], which we term *HH square LDC*, to arbitrary rectangular size $T \times M$ with rate-M, for both $T \geq M$ and $T < M$. These are termed uniform linear dispersion codes (U-LDC). First, a crucial unitary property of arbitrary U-LDC encoding matrices is established. Following this, a connection is made between this unitary property and the traceless minimal non-orthogonality criterion for space-time codes. Based on U-LDC, this paper proposes trace-orthonormal uniform LDC (TON-U-LDC) for $2L$ and $4L$ input symbols, for integers $L > 0$. Unlike U-LDC, the dispersion matrices for the real and imaginary components of source symbols in TON-U-LDC may differ. While TON-U-LDC achieves maximal symbolwise diversity order in block fading channels, in rapid fading channels, the symbol-wise time diversity order of $T \times M$ TON-U-LDC $2L$ is shown to be min(T,$2M$). Finally, in comparison to a number of other codes, TON-U-LDC are shown to perform effectively in both block and rapid fading channels.

The following notation is used: $(\cdot)^{\mathcal{T}}$ and $(\cdot)^{\mathcal{H}}$ for matrix transpose and matrix transpose conjugate, respectively, $\delta(\cdot)$ for Kronecker delta, $\mathrm{Tr}(\cdot)$ for matrix trace, j for $\sqrt{-1}$, $\mathbf{I}_K$ for a $K \times K$ identity matrix, and $\mathbf{0}_{A\times B}$ for an $A \times B$ all-zero matrix.

Manuscript received December 23, 2007; revised July 16, 2009; accepted September 10, 2009. The associate editor coordinating the review of this letter and approving it for publication was A. Nosratinia.

The authors are with the Department of Electrical and Computer Engineering, Queen's University, Kingston, Canada, K7L3N6 (e-mail: wujs@ieee.org; steven.blostein@queensu.ca).

This paper was presented in part at Globecom 2005. This research was supported by Natural Sciences and Engineering Research Council of Canada Discovery Grant 41731.

Digital Object Identifier 10.1109/TWC.2010.02.090065

II. LDC Encoding in Matrix Form

Assume that uncorrelated input bits have been modulated using complex-valued symbols chosen from an arbitrary constellation. A $T \times M$ LDC matrix codeword, $\mathbf{S}_{LDC}$, is transmitted from M transmit channels and occupies T channel uses and encodes Q source symbols [1]. The matrix codeword $\mathbf{S}_{LDC}$ is expressed as

$$\mathbf{S}_{LDC} = \sum_{q=1}^{Q} \alpha_q \mathbf{A}_q + j\beta_q \mathbf{B}_q, \tag{1}$$

where $\mathbf{S}_{LDC} \in C^{T\times M}$, and $\mathbf{A}_q \in C^{T\times M}, \mathbf{B}_q \in C^{T\times M}, q = 1, ..., Q$ are called dispersion matrices, the data symbol constellation is $s_q = \alpha_q + j\beta_q, q = 1, ..., Q$. An alternative dispersion matrix definition [1] is

$$\mathbf{S}_{LDC} = \sum_{q=1}^{Q} s_q \mathbf{C}_q + s_q^* \mathbf{D}_q, \tag{2}$$

where $\mathbf{C}_q = \frac{1}{2}(\mathbf{A}_q + \mathbf{B}_q)$ and $\mathbf{D}_q = \frac{1}{2}(\mathbf{A}_q - \mathbf{B}_q)$, $q = 1, ..., Q$. This paper considers the cases of both $\mathbf{A}_q = \mathbf{B}_q$ and $\mathbf{A}_q \neq \mathbf{B}_q$, where $q = 1, ..., Q$. For the case of $\mathbf{A}_q = \mathbf{B}_q$, $q = 1, ..., Q$, reordering $\mathbf{S}_{LDC}$ by using vec(.) operation, we obtain $\mathrm{vec}(\mathbf{S}_{LDC}) = \mathbf{G}_{LDC}\mathbf{s}$, where $\mathbf{G}_{LDC} = [\mathrm{vec}(\mathbf{A}_1), ..., \mathrm{vec}(\mathbf{A}_Q)]$ is LDC encoding matrix, and $\mathbf{s} = [s_1, ..., s_Q]^{\mathcal{T}}$.

The data symbol coding rate of LDC is defined as [1]

$$R_{LDC}^{sym} = \frac{Q}{T}. \tag{3}$$

III. Uniform Linear Dispersion Codes (U-LDC)

A. U-LDC Construction

Denote $\mathcal{D}_K = \text{diag}\left(1, e^{j\frac{2\pi}{K}}, ..., e^{j\frac{2\pi(K-1)}{K}}\right)$. Denote $\mathbf{\Pi}_K$ as a matrix of size $K \times K$ with zeros except $[\mathbf{\Pi}_K]_{a,a-1} = 1, a = 2, ..., K$ and $[\mathbf{\Pi}_K]_{K,K} = 1$. Denote $\mathbf{X} = \left[\mathbf{I}_T, \mathbf{Z}_{T\times(M-T)}\right]$ where $\mathbf{Z}_{T\times(M-T)}$ is a zero matrix.

1) The case of $T \leq M$: Denote $\mathbf{\Gamma} = \mathbf{X}$. The $T \times M$ U-LDC dispersion matrices are:

$$\mathbf{A}_{M(k-1)+l} = \mathbf{B}_{M(k-1)+l} = \frac{1}{\sqrt{T}} \left[\mathcal{D}_T\right]^{k-1} \mathbf{\Gamma} \left[\mathbf{\Pi}_M\right]^{l-1}, \quad (4)$$

where $k = 1, ..., T$ and $l = 1, ..., M$.

2) The case of $T > M$: Denote $\mathbf{\Gamma} = \mathbf{X}^{\mathcal{T}}$. The $T \times M$ U-LDC dispersion matrices are:

$$\mathbf{A}_{M(k-1)+l} = \mathbf{B}_{M(k-1)+l} = \frac{1}{\sqrt{T}} \left[\mathbf{\Pi}_T\right]^{k-1} \mathbf{\Gamma} \left[\mathcal{D}_M\right]^{l-1}, \quad (5)$$

where $k = 1, ..., T$ and $l = 1, ..., M$.

B. U-LDC Properties

Uniform linear dispersion codes of arbitrary size simultaneously hold the following two properties:

1) Unitary encoding matrix:

Property 1: For uniform linear dispersion codes with arbitrary $T \times M$ size dispersion matrices $\mathbf{A}_q, q = 1, ..., TM$, the encoding matrix $\mathbf{G}_{LDC} = [\text{vec}(\mathbf{A}_1), ..., \text{vec}(\mathbf{A}_{TM})]$ is unitary.

A proof of Property 1 is provided in Appendix A. Property 1 implies other desirable properties mentioned in Appendix A. According to [1], [2], Property 1 ensures capacity and energy-optimality in block-fading space-time channels. For $T \geq M$, U-LDC meets the more restrictive constraint

$$\left[\mathbf{A}_q\right]^{\mathcal{H}} \mathbf{A}_q = \frac{1}{M}\mathbf{I}_M. \quad (6)$$

This unitary property ensures arbitrary size $T \times M$ dispersion matrices $\mathbf{A}_q, q = 1, ..., TM$ satisfy the traceless minimal non-orthogonality criterion for block quasi-static fading channels [5]:

$$\text{Tr}\left[\left[\mathbf{A}_{q_1}\right]^{\mathcal{H}} \mathbf{A}_{q_2}\right] = \text{Tr}\left[\mathbf{A}_{q_1} \left[\mathbf{A}_{q_2}\right]^{\mathcal{H}}\right] = 0, \quad (7)$$

for any $1 \leq q_1 \neq q_2 \leq TM$. The traceless minimal non-orthogonality criterion [5] has been related to the error union bound (EUB) [6], [7]. In [7], Tirkkonen and Kokkonen have proven that (7) minimizes the dominant self-interference related to EUB. To the authors' knowledge, this paper is the first to establish the link between trace-based criteria and unitary criteria for space-time coding designs.

2) Symbolwise diversity [5], [8]: is a special case of full diversity since protection against single-symbol errors is a necessary condition for full diversity's protection against multiple-symbol errors.

Property 2: Uniform linear dispersion codes of arbitrary size $T \times M$ dispersion matrices $\mathbf{A}_q, q = 1, ..., TM$ achieve symbolwise diversity order $r = \min\left\{\text{rank}\left(\mathbf{A}_q\right), q = 1, ..., Q\right\} = \min\{M, T\}$.

Property 2 can be easily proven. To accommodate the requirement of arbitrary size, only full symbolwise diversity is guaranteed, while both [3] and [4] consider the achievement of full diversity.

IV. Construction of Trace Orthonormal Linear Dispersion Codes

A. Introduction

Using $\{\mathbf{C}_q, \mathbf{D}_q\}$ as dispersion matrices, trace orthonormal linear dispersion codes (TON-LDC) of size $KM \times M$ were proposed in [3]. TON-LDC may achieve the lower bound of the worst-case pairwise error probability of the maximum likelihood detector [3]. We remark that U-LDC is also TON-LDC with $\mathbf{A}_q = \mathbf{B}_q$, where $q = 1, ..., Q$. However, in this section, we aim to construct TON-LDC with $\mathbf{A}_q \neq \mathbf{B}_q$, where $q = 1, ..., Q$.

If the source data symbols are energy-normalized as $\mathbb{E}(\|s_q\|^2) = 1$, then the general conditions of TON-LDC [3] can be re-expressed as follows:

1) For the case of $T > M$:

$$\left[\mathbf{C}_q\right]^{\mathcal{H}} \mathbf{C}_q + \left[\mathbf{D}_q\right]^{\mathcal{H}} \mathbf{D}_q = \frac{T}{Q}\mathbf{I}_M. \quad (8)$$

2) For the cases of both $T > M$ and $T \leq M$:

$$\text{a) Tr}\left(\mathbf{C}_q \left[\mathbf{C}_p\right]^{\mathcal{H}} + \mathbf{D}_p \left[\mathbf{D}_q\right] \mathcal{H}\right) = \tfrac{MT}{Q}\delta(p-q) \quad (9)$$

$$\text{b) Tr}\left(\mathbf{D}_q \left[\mathbf{C}_p\right]^{\mathcal{H}} + \mathbf{D}_p \left[\mathbf{C}_q\right]^{\mathcal{H}}\right) = 0 \quad (10)$$

where $\mathbf{C}_p$ (or $\mathbf{C}_q$) and $\mathbf{D}_p$ (or $\mathbf{D}_q$) for $\{p, q\} = 1, ..., Q$ are defined in Section II.

We remark that in [9], TON-LDC may have the additional condition

$$\left[\mathbf{D}_q\right]^{\mathcal{H}} \mathbf{C}_q + \left[\mathbf{C}_q\right]^{\mathcal{H}} \mathbf{D}_q = \mathbf{0}, \quad (11)$$

where $q = 1, ..., Q$. However, a well-performing TON-LDC may not necessarily satisfy condition (11), e.g., see the 2×2 optimal design on page 626 of [9]. In the following, we only assume conditions (8), (9), and (10).

B. Construction of trace orthonormal linear dispersion codes for $Q = 2L$

1) General procedure: The following provides a general procedure to construct TON-LDC for the case of an even number of data source symbols. In this construction procedure, the new TON-LDC dispersion matrices, denoted by $\left\{\mathbf{A}_q^{(2)}, \mathbf{B}_q^{(2)}, \mathbf{C}_q^{(2)}, \mathbf{D}_q^{(2)}\right\}$, are constructed from an existing LDC denoted by matrices $\left\{\mathbf{A}_q^{(1)}, \mathbf{B}_q^{(1)}, \mathbf{C}_q^{(1)}, \mathbf{D}_q^{(1)}\right\}$, where $\mathbf{A}_q^{(1)} = \mathbf{B}_q^{(1)}$. Recall from Eq. (2) that $\mathbf{C}_q^{(1)}$ and $\mathbf{D}_q^{(1)}$ are functions of $\mathbf{A}_q^{(1)}$ and $\mathbf{B}_q^{(1)}$. In the TON-LDC, however, $\mathbf{A}_q^{(2)} \neq \mathbf{B}_q^{(2)}$.

Proposition 1: Consider a linear dispersion code with encoding matrix $\mathbf{G}_{LDC}^{(1)} = \left[\text{vec}(\mathbf{A}_1^{(1)})...\text{vec}(\mathbf{A}_Q^{(1)})\right]$ for Q data symbols as defined in Section II, where $\mathbf{A}_q^{(1)}, q = 1, ..., Q$, where Q is an even number, are the corresponding dispersion matrices of size $T \times M$. Assume the following holds for $p = 1, ..., Q$ and $q = 1, ..., Q$:

(i) *For the case of $T > M$,*

$$\left[\mathbf{A}_q^{(1)}\right]^{\mathcal{H}} \mathbf{A}_q^{(1)} = \frac{T}{Q}\mathbf{I}_M. \quad (12)$$

(ii) *For both cases of $T > M$ and $T \leq M$,*

$$\left[\mathrm{vec}(\mathbf{A}_p^{(1)})\right]^{\mathcal{H}} \mathrm{vec}(\mathbf{A}_q^{(1)}) = \mathrm{Tr}\left(\mathbf{A}_q^{(1)}\left[\mathbf{A}_p^{(1)}\right]^{\mathcal{H}}\right) = \frac{TM}{Q}\delta(p-q). \quad (13)$$

The codes are constructed by defining the following matrices:

(i) *For $q = 1, ..., Q/2$,*

$$\mathbf{C}_q^{(2)} = \frac{1}{\sqrt{2}}\mathbf{A}_q^{(1)}, \quad \mathbf{D}_q^{(2)} = \frac{1}{\sqrt{2}}e^{j\mu}\mathbf{A}_{q+(Q/2)}^{(1)}. \quad (14)$$

(ii) *For $q = ((Q/2)+1), ..., Q$,*

$$\mathbf{C}_q^{(2)} = \mathbf{A}_{\sigma(q-(Q/2))}^{(1)}, \quad \mathbf{D}_q^{(2)} = -e^{j\mu}\mathbf{A}_{(Q/2)+\sigma(q-(Q/2))}^{(1)}. \quad (15)$$

where μ is an arbitrary real number, and $\sigma(a)$ is an arbitrary fixed permutation of $a = 1, ..., Q/2$.

Then $\mathbf{C}_q^{(2)}$ and $\mathbf{D}_q^{(2)}$ consist of a set of dispersion matrices, as defined in Section II, of trace orthonormal linear dispersion codes.

The proof of Proposition 1 is omitted due to space limitations. Proposition 1 demonstrates that there are an infinite number of TON-LDC constructions which satisfy conditions (8), (9), and (10). Proposition 1 can be used not only for constructing rate-M TON-LDC, but also for constructing low-rate TON-LDC. Unlike in [9], Proposition 1 provides flexible choices of codeword sizes $T \times M$ and only requires that the number of source data symbols be even.

2) A special subclass of codes - TON-U-LDC-$2L$: Note that U-LDC are consistent with the assumptions of Proposition 1 in the case of $Q = TM$, where Q is even, i.e., $Q = 2L$. We propose to use dispersion matrices of U-LDC as $\left\{\mathbf{A}_q^{(1)}\right\}$ and apply the procedure in Section IV-B1 to generate trace-orthonormal uniform linear dispersion codes, denoted as TON-U-LDC-$2L$.

To gain more insight into TON-U-LDC-$2L$, it is useful to calculate dispersion matrices $\mathbf{A}_q^{(2)}$ and $\mathbf{B}_q^{(2)}$,

$$\mathbf{A}_q^{(2)} = \mathbf{C}_q^{(2)} + \mathbf{D}_q^{(2)} = \begin{cases} \mathbf{A}_q^{(1)} + e^{j\mu}\mathbf{A}_{q+(Q/2)}^{(1)}, & q = 1, ..., Q/2 \\ \mathbf{A}_{\sigma(q-(Q/2))}^{(1)} - e^{j\mu}\mathbf{A}_{(Q/2)+\sigma(q-(Q/2))}^{(1)}, & q = ((Q/2)+1), ..., Q \end{cases}$$

$$\mathbf{B}_q^{(2)} = \mathbf{C}_q^{(2)} - \mathbf{D}_q^{(2)} = \begin{cases} \mathbf{A}_q^{(1)} + e^{j\mu}\mathbf{A}_{q+(Q/2)}^{(1)}, & q = 1, ..., Q/2 \\ \mathbf{A}_{\sigma(q-(Q/2))}^{(1)} + e^{j\mu}\mathbf{A}_{(Q/2)+\sigma(q-(Q/2))}^{(1)}, & q = ((Q/2)+1), ..., Q \end{cases}$$

It is easy to verify $\mathrm{rank}(\mathbf{A}_q^{(2)}) = \mathrm{rank}(\mathbf{B}_q^{(2)}) = \min\{M, T\}$, and thus having real and imaginary components per symbol achieves full component-wise diversity.

Denote $\mathbf{c}_t = \left[c_t^1, ..., c_t^M\right]$ and $\mathbf{b}_t = \left[b_t^1, ..., b_t^M\right]$, $1 \leq t \leq T$. The probability of transmitting $\mathcal{C} = \left[[\mathbf{c}_1]^{\mathcal{T}}, ..., [\mathbf{c}_T]^{\mathcal{T}}\right]^{\mathcal{T}}$ and deciding in favor of $\mathcal{B} = \left[[\mathbf{b}_1]^{\mathcal{T}}, ..., [\mathbf{b}_T]^{\mathcal{T}}\right]^{\mathcal{T}}$ at the maximum-likelihood decoder is well approximated by [10]

$$P(\mathcal{C} \to \mathcal{B}) \leq \prod_{t \in \mathcal{V}(\mathcal{C},\mathcal{B})} \left(|\mathbf{c}_t - \mathbf{b}_t|^2 \rho/4\right)^{-N}, \quad (16)$$

where ρ is symbol signal-to-noise-ratio (SNR), $\mathcal{V}(\mathcal{C},\mathcal{B})$ denotes the set of time instances $1 \leq t \leq T$ such that $|\mathbf{c}_t - \mathbf{b}_t| \neq 0$ and $|\mathcal{V}(\mathcal{C},\mathcal{B})|$ denotes the number of elements of $\mathcal{V}(\mathcal{C},\mathcal{B})$. Thus, the diversity achieved is $N|\mathcal{V}(\mathcal{C},\mathcal{B})|$ for space-time rapid fading channels [10].

Note that the code construction of TON-U-LDC-$2L$ is quite flexible, and a general analysis of $N|\mathcal{V}(\mathcal{C},\mathcal{B})|$ is difficult. However, we introduce the concepts of symbol and (real and imaginary) component-wise diversity $|\mathcal{V}(\mathcal{C},\mathcal{B})|$ for a single input source symbol or component error as

$$|\mathcal{V}(\mathcal{C},\mathcal{B})|_s = \min\left\{ |\mathcal{V}(\mathcal{C},\mathcal{B})| \;\middle|\; \begin{array}{c} s_q^{\mathcal{C}} = s_q^{\mathcal{B}}, q = 1, ..., Q, \\ \text{except } \exists k, 1 \leqslant k \leqslant Q, \\ s_k^{\mathcal{C}} \neq s_k^{\mathcal{B}} \end{array} \right\}$$

and

$$|\mathcal{V}(\mathcal{C},\mathcal{B})|_c = \min\left\{ |\mathcal{V}(\mathcal{C},\mathcal{B})| \;\middle|\; \begin{array}{c} s_q^{\mathcal{C}} = s_q^{\mathcal{B}}, q = 1, ..., Q, \\ \text{except } \exists k, 1 \leqslant k \leqslant Q \\ \text{either } \alpha_k^{\mathcal{C}} = \alpha_k^{\mathcal{B}}, \beta_k^{\mathcal{C}} \neq \beta_k^{\mathcal{B}} \\ \text{or } \alpha_k^{\mathcal{C}} \neq \alpha_k^{\mathcal{B}}, \beta_k^{\mathcal{C}} = \beta_k^{\mathcal{B}} \end{array} \right\},$$

respectively, where $\left\{s_q^{\mathcal{C}} = \alpha_q^{\mathcal{C}} + j\beta_q^{\mathcal{C}}\right\}$ and $\left\{s_q^{\mathcal{B}} = \alpha_q^{\mathcal{B}} + j\beta_q^{\mathcal{B}}\right\}$ are source symbol sequences for $\mathcal{B}$ and $\mathcal{C}$, respectively.

For $T \geq 2M$, it can be verified that TON-U-LDC-$2L$ always achieves $|\mathcal{V}(\mathcal{C},\mathcal{B})|_s = |\mathcal{V}(\mathcal{C},\mathcal{B})|_c = 2M$, while the TON-LDC proposed in [9] only achieves $|\mathcal{V}(\mathcal{C},\mathcal{B})|_s = |\mathcal{V}(\mathcal{C},\mathcal{B})|_c = M$. Further, for $T \leq 2M$, $|\mathcal{V}(\mathcal{C},\mathcal{B})|_s = |\mathcal{V}(\mathcal{C},\mathcal{B})|_c = T$ is guaranteed to hold for TON-U-LDC-$2L$.

C. Construction of trace orthonormal linear dispersion codes for $Q = 4L$

Using dispersion matrices of U-LDC as $\left\{\mathbf{A}_q^{(1)}\right\}$, another set of dispersion matrices of trace orthonormal uniform linear dispersion codes for $Q = 4L$, denoted as TON-U-LDC $4L$, may be constructed as follows:

1) For $q = 1, ..., Q/4$,

$$\mathbf{C}_q^{(2)} = \frac{1}{\sqrt{2}}\mathbf{A}_q^{(1)}, \mathbf{D}_q^{(2)} = \frac{1}{\sqrt{2}}\mathbf{A}_{q+(Q/2)}^{(1)}. \quad (17)$$

2) For $q = 1 + (Q/4), ..., Q/2$,

$$\mathbf{C}_q^{(2)} = \frac{1}{\sqrt{2}}e^{j\mu}\mathbf{A}_{\tau(q-(Q/4))+(Q/4)}^{(1)}, \quad \mathbf{D}_q^{(2)} = \frac{1}{\sqrt{2}}e^{j\mu}\mathbf{A}_{\tau(q-(Q/4))+(3Q/4)}^{(1)}. \quad (18)$$

3) For $q = ((Q/2)+1), ..., 3Q/4$,

$$\mathbf{C}_q^{(2)} = \frac{1}{\sqrt{2}}\mathbf{A}_{q-(Q/2)}^{(1)}, \mathbf{D}_q^{(2)} = \frac{(-1)}{\sqrt{2}}\mathbf{A}_q^{(1)}. \quad (19)$$

4) For $q = 1 + (3Q/4), ..., Q$,

$$\mathbf{C}_q^{(2)} = \frac{1}{\sqrt{2}} e^{j\mu} \mathbf{A}^{(1)}_{\tau(q-(3Q/4))+(Q/4)},$$
$$\mathbf{D}_q^{(2)} = \frac{1}{\sqrt{2}} e^{j\mu} \mathbf{A}^{(1)}_{\tau(q-(3Q/4))+(3Q/4)}. \quad (20)$$

In the above, μ is an arbitrary real-valued constant and $\tau(a)$ is an arbitrary fixed permutation of $a = 1, ..., Q/4$.

D. Remarks

Note that a specific permutation determines a specific TON-U-LDC construction, and thus both TON-U-LDC 2L and 4L are a set of codes due to allowance of different permutations. According to our experiences, TON-U-LDC under different permutations approach similar performance for 2L and 4L cases, respectively.

Note that a specific permutation determines a specific TON-U-LDC construction. The TON-U-LDC 2L and 4L above each refer to a set of codes, allowing for different permutations. The authors have observed, however, that the performances of TON-U-LDC 2L and 4L are not affected significantly by these different permutations.

V. Performance

In this section, under the same spectral efficiency (bits per time channel use), we compare the U-LDC and TON-U-LDC codes with other well-known full-diversity rate-M designs: HH [1], TAST [11], FDFR [4], TON-LDC [3], [9], Golden codes [12], Heath-Paulraj (HP) codes [2], and Gohary-Davidson (GD) codes [13]. In the comparisons, MIMO flat fading channels are assumed. The matrix channel is assumed to be constant over a block of an integer number of symbol time slots, which we denote as the channel change interval (CCI), and changing independently and identically distributed between blocks. Data symbols use 4-QAM modulation in all simulations. The numbers of transmit and receive antennas are M and N, respectively, where $M = N$. Each LDC codeword is of size $T \times M$. Maximum likelihood decoding is performed at the receiver. Average symbol SNR per receive antenna is reported in all figures.

As shown in Figs. 1 and 2, in $M = N = 2$ MIMO block fading channels ($CCI = 4$), TON-U-LDC $2L$ of size 4 under $\mu = \frac{1}{4}\pi$ and permutation $\sigma(a)$ = $(Q/2)$ - a+1 performs second-best, while, in rapid fading channels ($CCI = 1$), TON-U-LDC $2L$ and TON-U-LDC $4L$ (under $\mu = \frac{1}{4}\pi$ and permutation $\tau(a) = a$) outperform the others.

As shown in Figs. 3 and 4, in $M = N = 3$ MIMO block fading channels ($CCI = 4$), U-LDC of size 5×3 performs the best, TON-U-LDC $2L$ of size 4×3 under $\mu = \frac{2}{3}\pi$ and permutation $\sigma(a) = a$ performs second best, while, in rapid fading channels ($CCI = 1$), TON-U-LDC $2L$ again performs the best.

Fig. 5 illustrates the effects of μ on the performance of TON-U-LDC $2L$ under permutation $\sigma(a) = (Q/2) - a + 1$ in block fading channels at 15dB SNR. However, in the case of rapid fading channels ($CCI = 1$), it has been observed that the performance of TON-U-LDC 2L is insensitive to the choice of μ.

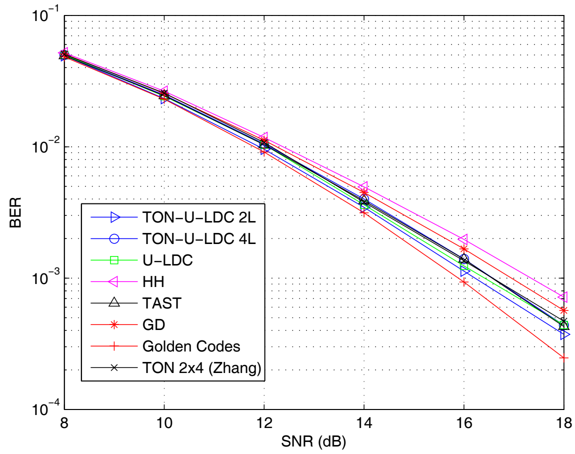

Fig. 1. BER performance comparison in space time block fading channels, $M = N = 2$, $CCI = 4$

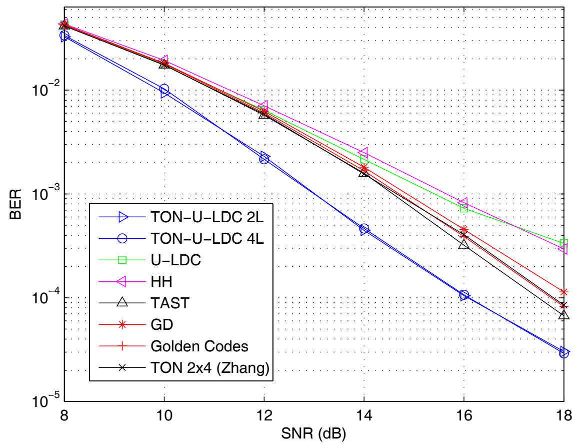

Fig. 2. BER performance comparison in space-time rapid fading channels, $M = N = 2$, $CCI = 1$

In summary, it can be observed that even though U-LDC and TON-U-LDC are not claimed to possess full diversity, they are able to outperform several well-known full diversity codes in the literature in certain cases. Although U-LDC, TON-U-LDC $2L$, and TON-U-LDC $4L$ have been compared with different space-time dimensions, we do not claim that they have superior performance over all other codes of arbitrary size. Rather, we conclude that U-LDC, TON-U-LDC $2L$, and TON-U-LDC $4L$ are all of flexible size, mathematically tractable, and possess desirable properties as discussed in Sections III and IV. We remark that the performance of U-LDC of larger dimensions for the case of MIMO-OFDM channels appears in [14].

In this paper, channels are assumed to be uncorrelated. In the future, code design for correlated channels could be studied.

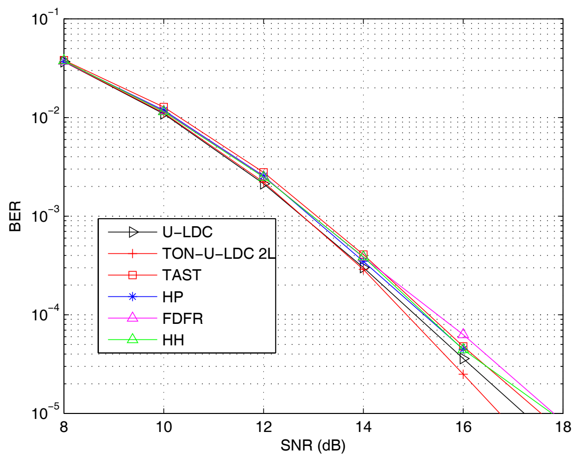

Fig. 3. BER performance comparison in space-time block fading channels, $M = N = 3$, $CCI = 60$

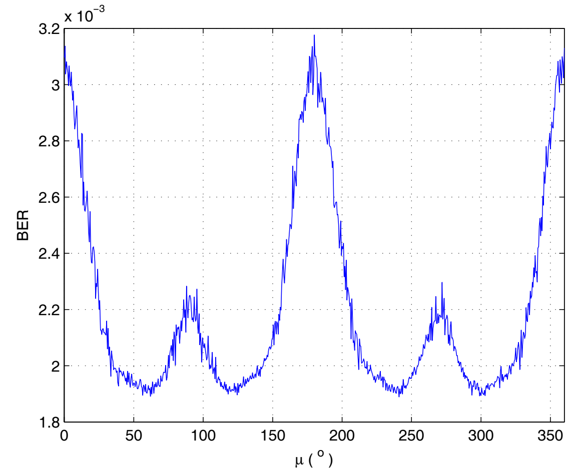

Fig. 5. Effects of μ on the performance of TON-U-LDC 2L in space-time block fading channels, $M = N = 2$, $T = 4$,, $CCI = 60$

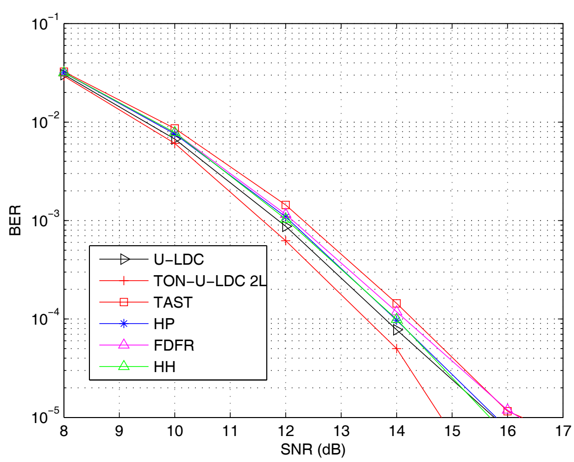

Fig. 4. BER performance comparison in space-time rapid fading channels, $M = N = 3$, $CCI = 1$

Appendix A
A Unified Proof of the Unitary Property

Proof: Consider $p = M(k_p - 1) + l_p$ and $q = M(k_q - 1) + l_q$, where $1 \le \{k_p, k_q\} \le T$ and $1 \le \{l_p, l_q\} \le M$. Denote

$$\Delta_{p,q} = \mathrm{Tr}\left(\mathrm{vec}(\mathbf{A}_q)\left[\mathrm{vec}(\mathbf{A}_p)\right]^{\mathcal{H}}\right) = \left[\mathrm{vec}(\mathbf{A}_p)\right]^{\mathcal{H}}\mathrm{vec}(\mathbf{A}_q) = \mathrm{Tr}\left(\left[\mathbf{A}_p\right]^{\mathcal{H}}\mathbf{A}_q\right).$$

1) The case of $T \le M$:
The following always holds for U-LDC,

$$\Delta_{p,q} = \frac{1}{T}\mathrm{Tr}\left(\left[\mathbf{\Pi}_M^{l_p-1}\right]^{\mathcal{H}}[\mathbf{\Gamma}]^{\mathcal{H}}\left[\mathcal{D}_T^{k_p-1}\right]^{\mathcal{H}}\mathcal{D}_T^{k_q-1}\mathbf{\Gamma}{\mathbf{\Pi}_M}^{l_q-1}\right) = \frac{1}{T}\mathrm{Tr}\left(\left([\mathbf{\Pi}_M]^{\mathcal{T}}\right)^{l_p-1}\mathbf{C}{\mathbf{\Pi}_M}^{l_q-1}\right),$$

where

$$\mathbf{C} = [\mathbf{\Gamma}]^{\mathcal{T}}\left[\mathcal{D}_T^{k_p-1}\right]^{\mathcal{H}}\mathcal{D}_T^{k_q-1}\mathbf{\Gamma} = \begin{pmatrix} \mathcal{D}_T^{k_q-k_p} & \mathbf{0}_{T\times(M-T)} \\ \mathbf{0}_{(M-T)\times T} & \mathbf{0}_{(M-T)\times(M-T)} \end{pmatrix}.$$

Note that the above proof uses the following facts

a) $\mathbf{\Pi}_M$ and $\mathbf{\Gamma}$ is real matrix, thus $[\mathbf{\Pi}_M]^{\mathcal{H}} = [\mathbf{\Pi}_M]^{\mathcal{T}}$ and $[\mathbf{\Gamma}]^{\mathcal{H}} = [\mathbf{\Gamma}]^{\mathcal{T}}$,

b) $\left[e^{j\theta_p}\right]^* e^{j\theta_q} = e^{-j\theta_p}e^{j\theta_q} = e^{j(\theta_q-\theta_p)}$.

Also, note that $\mathbf{\Pi}_M^{\mathcal{T}}\mathbf{C}$ rotates the rows of $\mathbf{C}$ upward, and $\mathbf{C}\mathbf{\Pi}_M$ rotates the columns of $\mathbf{C}$ leftward. So, $\mathbf{\Pi}_M^{\mathcal{T}}\mathbf{C}\mathbf{\Pi}_M$ just rotates the diagonal elements of $\mathbf{C}$ from top left corner to bottom right corner as $\mathbf{C}$ is a diagonal matrix and hence, $\mathrm{Tr}\left(\mathbf{\Pi}_M^{\mathcal{T}}\mathbf{C}\mathbf{\Pi}_M\right) = \mathrm{Tr}\left(\mathbf{C}\right)$. Also,

$$\mathrm{Tr}\left(\left([\mathbf{\Pi}_M]^{\mathcal{T}}\right)^{l_p-1}\mathbf{C}{\mathbf{\Pi}_M}^{l_q-1}\right) = \begin{cases} \mathrm{Tr}(\mathbf{C}), & l_p = l_q \\ 0, & l_p \ne l_q \end{cases}$$

Thus,

$$\Delta_{p,q} = \begin{cases} \mathrm{Tr}(\mathbf{C}), & l_p = l_q \text{ and } k_p = k_q,\ i.e., p = q \\ 0, & l_p \ne l_q \end{cases}$$

2) The case of $T > M$: Using $\mathrm{Tr}\left(\mathbf{XY}\right) = \mathrm{Tr}\left(\mathbf{YX}\right)$,

$$\Delta_{p,q} = \frac{1}{M}\mathrm{Tr}\left(\left[\mathcal{D}_M^{l_p-1}\right]^{\mathcal{H}}\mathbf{\Gamma}\left[\mathbf{\Pi}_T^{k_p-1}\right]^{\mathcal{H}}\mathbf{\Pi}_T^{k_q-1}\mathbf{\Gamma}{\mathcal{D}_M}^{l_q-1}\right) = \frac{1}{M}\mathrm{Tr}\left(\mathbf{\Pi}_T^{k_q-1}\mathbf{\Gamma}{\mathcal{D}_M}^{l_q-1}\left[\mathcal{D}_M^{l_p-1}\right]^{\mathcal{H}}[\mathbf{\Gamma}]^{\mathcal{T}}\left[\mathbf{\Pi}_T^{k_p-1}\right]^{\mathcal{T}}\right).$$

The rest of the proof for the case of $T > M$ is similar to that of the case of $T \le M$.

■

REFERENCES

[1] B. Hassibi and B. M. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. Inf. Theory*, vol. 48, no. 7, pp. 1804-1824, July 2002.

[2] R. W. Heath and A. J. Paulraj, "Linear dispersion codes for MIMO systems based on frame theory," *IEEE Trans. Signal Process.*, vol. 50, no. 10, pp. 2429-2441, Oct. 2002.

[3] J.-K. Zhang, K. M. Wong, and T. N. Davidson, "Information lossless full rate full diversity cyclotomic linear dispersion codes," in *Proc. IEEE ICASSP 2004*, vol. 4, May 2004, pp. 465-468.

[4] X. Ma and G. B. Giannakis, "Full-diversity full-rate complex-field space-time coding," *IEEE Trans. Signal Process.*, vol. 51, no. 11, pp. 2917-2930, Nov. 2003.

[5] O. Tirkkonen and A. Hottinen, "Improved MIMO performance with non-orthogonal space-time block codes," in *Proc. IEEE Global Telecommun. Conf. (Globecom) 2001*, vol. 2, Nov. 2001, pp. 1122-1126.

[6] S. Sandhu and A. Paulraj, "Union bound on error probability of linear space-time block codes," in *Proc. IEEE ICASSP 2001*, vol. 4, May 2001, pp. 2473-2476.

[7] O. Tirkkonen and M. Kokkonen, "Interference, information and performance in linear matrix modulation," in *Proc. IEEE PIMRC 2004*, vol. 1, Sept. 2004, pp. 27-32.

[8] O. Tirkkonen and A. Hottinen, "Maximal symbolwise diversity in non-orthogonal space-time block codes," in *Proc. IEEE Int. Symposium Inform. Theory (ISIT) 2001*, June 2001, pp. 197-197.

[9] J.-K. Zhang, J. Liu, and K. M. Wong, "Trace-orthonormal full-diversity cyclotomic space-time codes," *IEEE Trans. Signal Process.*, vol. 55, no. 2, pp. 618-630, Feb. 2007.

[10] V. Tarokh, N. Seshadri, and A. Calderbank, "Space-time codes for high data rate wireless communications: performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, pp. 744-765, Mar. 1998.

[11] H. E. Gamal and M. O. Damen, "Universal space-time coding," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1097-1119, May 2003.

[12] J. C. Belfiore, G. Rekaya, and E. Viterbo, "The Golden code: A 2 x 2 full-rate space-time code with non-vanishing determinants," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1432-1436, Apr. 2005.

[13] R. H. Gohary and T. N. Davidson, "Linear dispersion codes: asymptotic guidlines and their implementation," *IEEE Trans. Wireless Commun.*, vol. 4, no. 6, pp. 2892-2906, Nov. 2005.

[14] J. Wu and S. D. Blostein, "High-rate codes over space, time, and frequency," in *Proc. IEEE Global Telecommun. Conf. (Globecom) 2005*, vol. 6, Nov. 2005, pp. 3602-3607.