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Low Complexity MIMO Precoding Codebooks from Orthoplex Packings

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***Abstract*—A construction of Grassmannian packings related to representation theory is applied to build implementation-friendly MIMO precoding codebooks when the number of transmit antennas is a power of a prime number. Using chordal distance as a metric, some of the corresponding packings are optimal by meeting the orthoplex bound. The codebooks are multimodal and can be generated from a finite alphabet which is beneficial for hardware implementation. In addition, by using only some of the codewords, smaller packings satisfying the equal power per antenna constraint can be constructed. Optimality with reference to this constraint is shown by using a modification of Conway-Hardin-Sloane's spherical embedding of the Grassmann manifold for equal per-antenna codebooks.**

I. Introduction

Multiple-Input Multiple-Output (MIMO) wireless communications systems using linear precoding have been shown to achieve large capacity gains over traditional single-input-single-output (SISO) systems [1]. Full gains from precoding are achieved when the transmitter possesses perfect channel state information (CSI). Often, especially in frequency-division duplex systems, full CSI is not available at the transmitter. Limited feedback is a practical way to obtain partial CSI at the transmitter. In a codebook-based precoding transmission, the receiver transmits the index of a codeword from a pre-designed codebook shared with the transmitter.

The problem of designing precoding codebooks for transmission reduces to a discretization problem of the Grassmann manifold. The information rate of a limited feedback MIMO system has been shown to be closely approximated as a function of the distortion rate associated with a quantization on the Grassmann manifold with chordal distance as a metric [2]. Maximizing the minimum distance of the codebook is an appropriate design criterion to obtain Grassmannian codes with small distortion [2], [3]. Such codebooks are called Grassmannian packings. Because analytical construction of the optimum packing is possible only in very special cases, the available codebooks are mostly generated by computer searches using vector quantization algorithms such as Lloyd's algorithm [4], and alternating projection algorithms [5].

However, several benefits arise from analytical constructions. One benefit of having geometric insight into the codebooks, and a corresponding analytical handle on their design, is that suitable rotations may be found to simplify the codebook representation so that it can be realized with a set of different complex numbers with minimum cardinality. Besides being aesthetically pleasing, this is beneficial for implementation from two perspectives. First, hardware operations on complex numbers are performed with cordic rotators, which are complicated for arbitrary angles. Second, more numerous and more arbitrary rotations are more prone to implementation errors. Also, control on the codebooks may be required to select the way the codebooks map to antennas. For example, it may be desirable to have a codebook where some codewords are transmitted from one antenna only. Alternatively, it may be required to satisfy an equal power per-antenna constraint. Furthermore, precoding codebooks for different streams need to be separately stored in memory. Recent standards, e.g. 3GPP, promote structured finite-alphabet codebooks employing matrix concatenations eliminating the requirement to store multiple codebooks.

There has been renewed interest in decreasing implementation complexity of MIMO systems in order to short-track industry adoption [6]. The authors in [7] and [8] introduced structured codebooks having two main properties: (i) the alphabet of the codeboook elements is restricted to the quaternary alphabet $\{1, -1, \mathrm{i}, -\mathrm{i}\}$, $\mathrm{i} = \sqrt{-1}$, and (ii) a lower rank precoding matrix is a submatrix of a higher rank precoding matrix. Additionally, these codebooks are constructed from mutually unbiased bases guaranteeing the benefit of equal transmit-power from all antennas and for all transmission ranks. In [7], the authors show that the quaternary alphabet facilitates efficient codebook storage and codebook search. Storage is reduced further by deriving higher-rank codebooks from lower-rank codebooks. In [8], a similar codebook design was proposed for multiuser MIMO. The quaternary codebook alphabet is employed to reduce the complexity necessary at the receiver for choosing a codeword by enabling matrix multiplication to be performed only with conjugations and additions.

In this paper, we design MIMO precoding codebooks for cases where the number of transmit antennas is a power of a prime. Using the orthoplex bound for the complex Grassmannian, these codebooks are shown to be optimum complex packings. Some codebooks reach the bound with the maximum number of points. These codebooks can be generated from a finite alphabet and the cardinality can be decreased in order to meet an equal power per antenna constraint. Some of the constructions are shown to satisfy this constraint in an optimal manner using a modification of the Conway-Hardin-Sloane spherical embedding of the Grassmannian for equal power per-antenna codebooks. The codebook design proposed is based on a group-theoretic framework for packings in the real Grassmannian, provided in [9] for dimensions that are powers of 2. This framework is based on the properties of *Extraspecial 2-groups* [10]. We generalize this construction to the complex Grassmannian and for any power of a prime p, based on the extensions of this extraspecial 2-group in [10].

For $p = 2$, the construction presented here is a specific case of the constructions presented in [11], [12], [13]. In some cases analyzed in [11], [12], relevant for non-coherent MIMO systems, the codebooks' cardinality is greatly extended without decreasing the minimum distance when the codebooks are not optimal. In cases when the orthoplex bound is achieved with a maximum number of points, configurations cannot be extended without decreasing the minimum distance. For limited feedback MIMO precoding, a small number of information bits describing the channel quality can allow near optimal channel adaptation [14]. In this paper, we are thus looking for low cardinality codebooks with good distance and implementation properties but not necessarily extensively large codebooks. Our construction is closely related to mutually unbiased bases and the structured codebooks from [7] can be understood as a specific case of this construction.

II. System model, Notation and related spaces

We consider precoded transmission over a MIMO channel with N_t and N_r transmit and receive antennas, respectively. The signal model is $\mathbf{y} = \mathbf{H}\mathbf{W}\mathbf{x} + \mathbf{n}$, where $\mathbf{W}$ is a $N_t \times N_s$ precoding matrix, $\mathbf{x}$ is a $N_s \times 1$ vector of information symbols, and N_s is the number of streams. We concentrate on the properties of $\mathbf{W}$ related to steering the transmitted energy to the signal subspace of the receiver. Power allocation and orthogonalization of the streams are considered to be out of scope of precoding and instead form a part of the design of $\mathbf{x}$. Then, we have the following contraints on $\mathbf{W}$: the total transmit power is $\mathrm{Tr}[\mathbf{W}^H\mathbf{W}] = N_s$, which is equally shared among the symbols such that $\mathrm{diag}(\mathbf{W}^H\mathbf{W}) = \mathbf{1}_{N_s\times 1}$. We also consider constraining the codebook so that $\mathrm{diag}(\mathbf{W}\mathbf{W}^H) = N_s/N_t\,\mathbf{1}_{N_t\times 1}$ which is the *equal power per-antenna* constraint.

In [3], it was argued that the problem of designing a codebook of matrices $\mathbf{W}$ reduces to discretizing the Grassmann manifold. The complex Grassmann manifold $\mathrm{G}^{\mathbb{C}}_{n,p}$ is the set of all p-dimensional subspaces of $\mathbb{C}^n$. $\mathrm{G}^{\mathbb{C}}_{n,p}$ can be expressed as the quotient space of the Stiefel manifold and the unitary group: $\mathrm{G}^{\mathbb{C}}_{n,p} \cong \mathrm{V}^{\mathbb{C}}_{n,p}/\mathcal{U}_p$. The complex Stiefel manifold $\mathrm{V}^{\mathbb{C}}_{n,p}$ can be defined as the space of orthonormal rectangular matrices (with $p \leq n$):

$$\mathrm{V}^{\mathbb{C}}_{n,p} = \left\{ Y \in \mathbb{C}^{n\times p} \quad | \quad Y^H Y = I_p \right\}. \tag{1}$$

A point in the Grassmann manifold can thus be represented as the equivalence class of the $n \times p$ orthonormal matrices whose columns span the same space:

$$[Y] = \{ Y U_p \quad | \quad U_p \in \mathcal{U}_p \}. \tag{2}$$

We denote by $[Y]$, $[Z] \in \mathrm{G}^{\mathbb{C}}_{n,p}$ two subspaces of $\mathbb{C}^n$, where Y, $Z \in \mathrm{V}^{\mathbb{C}}_{n,p}$ are representative of their respective equivalence classes. The chordal distance is defined as [15]

$$d_c([Y],[Z]) = \frac{1}{\sqrt{2}} \| YY^H - ZZ^H \|_F. \tag{3}$$

Additionally, we introduce spaces related to the equal power per-antenna constraint:

Definition 1: The constrained complex Stiefel manifold $\mathrm{V}^{\mathbb{C},\mathrm{EP}}_{n,p}$ is defined as

$$\mathrm{V}^{\mathbb{C},\mathrm{EP}}_{n,p} = \left\{ Y \in \mathrm{V}^{\mathbb{C}}_{n,p} \quad | \quad \textstyle\sum_j |Y_{i,j}|^2 = p/n \quad \forall i \right\}, \tag{4}$$

and

Definition 2: The constrained complex Grassmann manifold, $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{n,p}$, is defined as $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{n,p} \cong \mathrm{V}^{\mathbb{C},\mathrm{EP}}_{n,p}/\mathcal{U}_p$.
Obviously $\mathrm{V}^{\mathbb{C},\mathrm{EP}}_{n,p} \subset \mathrm{V}^{\mathbb{C}}_{n,p}$, $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{n,p} \subset \mathrm{G}^{\mathbb{C}}_{n,p}$, and the notion of distance is preserved on the constrained Grassmann manifold.

III. Embedding and corresponding bounds

For each $[Y] \in \mathrm{G}^{\mathbb{C}}_{n,p}$, we associate the orthogonal projection from $\mathbb{C}^n$ to $[Y]$: $\Pi_Y = YY^H$. This projection is unique for every element of $\mathrm{G}^{\mathbb{C}}_{n,p}$ and independent of the equivalence class representative. The chordal distance can be rewritten as $d_c([Y],[Z]) = \frac{1}{\sqrt{2}} ||\Pi_Y - \Pi_Z||_F$.

The projector matrix is an $n \times n$ Hermitian idempotent matrix which lies in the space of Hermitian matrices with trace equal to p. This space has dimension $n^2 - 1$.

For the constrained Grassmannian $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{n,p}$, the associated projectors lie in the space of Hermitian matrices which have constant value p/n on the main diagonal. This space has dimension $n^2 - n$.

From Conway, Hardin and Sloane [16, Theroem 2] and its generalization to the complex case [17], we know that the complex Grassmannian manifold equipped with chordal distance, $(\mathrm{G}^{\mathbb{C}}_{n,p}, d_c)$, has an isometric embedding into a sphere of radius $\sqrt{\frac{p(n-p)}{2n}}$ in $\mathbb{R}^D$ with $D = n^2 - 1$. This result implies the following corollary which is a generalization of the result for real Grassmannian [15] to the complex case. Since a codebook in $\mathrm{G}^{\mathbb{C}}_{n,p}$ is isometric to a subset of a hypersphere, we can apply the Rankin bounds [16]:

Corollary 1: For a packing of N points in $\mathrm{G}^{\mathbb{C}}_{n,p}$ equipped with the chordal distance, the minimum distance among the elements of the packing is bounded by:

1) The simplex bound:
$$\delta^2 \leq \frac{p(n-p)}{n} \cdot \frac{N}{N-1} \tag{5}$$
which is achievable only if $N \leq D + 1 = n^2$.
2) The orthoplex bound: for $N > n^2$
$$\delta^2 \leq \frac{p(n-p)}{n} \tag{6}$$
which is achievable only if $N \leq 2D = 2(n^2 - 1)$.

Similar results can be provided for the constrained Grassmannian. We obtain a smaller embedding with the same radius as the one in [15], [17].

Proposition 1: The constrained equal power complex Grassmannian manifold, $(\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{n,p}, d_c)$, has an isometric embedding into a sphere of radius $\sqrt{\frac{p(n-p)}{2n}}$ in $\mathbb{R}^{D'}$ with $D' = n^2 - n$.

Accordingly, we obtain the following bounds on the minimum distance of a packing in the constrained Grassmannian:

Corollary 2: For a packing of N points in $(\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{n,p}, d_c)$, the minimum distance among the elements of the packing is bounded by:

1) The simplex bound:
$$\delta^2 \le \frac{p(n-p)}{n} \cdot \frac{N}{N-1} \tag{7}$$
which is achievable only if $N \le D' + 1 = n^2 - n + 1$.
2) The orthoplex bound: for $N > n^2 - n + 1$
$$\delta^2 \le \frac{p(n-p)}{n} \tag{8}$$
which is achievable only if $N \le 2D' = 2(n^2 - n)$.

The last embedding lies on an "equatorial" sphere $S^{D'-1}$ of the hypersphere S^{D-1}. Thus, only the ranges of the bounds are modified. An example is provided in Figure 1 for packing planes in four dimensions.

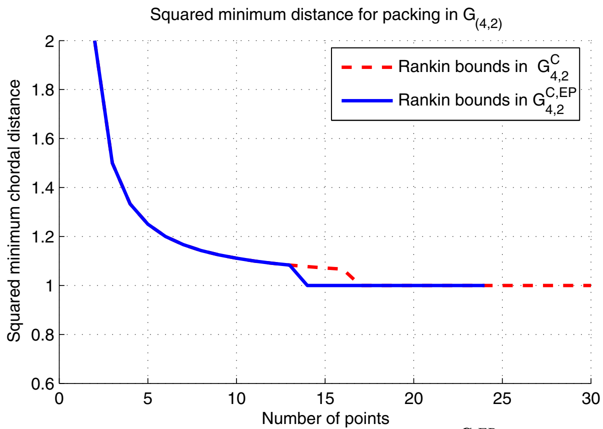

Fig. 1. Rankin bounds in $\mathrm{G}^{\mathbb{C}}_{4,2}$ and $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{4,2}$

IV. Packings from an Extraspecial p-Group

The following construction is a generalization of [9] to the complex case and any prime power. The construction follows from the method of associating a symplectic space to an extraspecial group [18, Th. 23.10]. The aim is to constuct packings of N_s-spaces in $\mathbb{C}^{N_t}$ where N_t and N_s are powers of a prime.

A. The extraspecial group

A finite group is a p-group if and only if its order is a power of p, where p is a prime number. Every element in a p-group has order a power of p. Given a prime p, a p-group P is said to be $extraspecial$ if its center $Z(P)$ is cyclic and if $P/Z(P)$ is elementary abelian [18, Ch. 8].

We are interested in dimensions $N_t = p^m$ that are powers of a prime p, with m a strictly positive integer. Let $\mathbb{F}_p^m$ denote an m-dimensional vector space over the field of integers modulo p, $\mathbb{F}_p$. We index the p^m elements e_u of the standard basis of $\mathbb{C}^{N_t}$, with the p^m elements $u \in \mathbb{F}_p^m$. Let ξ be a primitive pth root of unity in $\mathbb{C}$.

For $a, b \in \mathbb{F}_p^m$, define the following generalized Pauli matrices:
$$X(a) : e_u \to e_{u+a}\ , \quad Y(b) : e_u \to \xi^{b^{\mathrm{T}}u} e_u\ , \quad u \in \mathbb{F}_p^m$$
and the indicator function $\delta_p = 1$ if $p = 2$ and 0 otherwise. The groups $X = \langle X(a) \ \mid\ a \in \mathbb{F}_p^m \rangle$ and $Y = \langle Y(b) \ \mid\ b \in \mathbb{F}_p^m \rangle$ are isomorphic to the additive group $\mathbb{F}_p^m$. Then,
$$E := \{\mathrm{i}^{\alpha\delta_p}\xi^\lambda X(a)Y(b) \ \mid\ \lambda \in \mathbb{F}_p, \quad a, b \in \mathbb{F}_p^m, \alpha \in \mathbb{F}_2\}$$
is an extraspecial group of order $p^{2m+1+\delta_p}$ with center $Z(E) = \langle \mathrm{i}^{\delta_p}\xi I\rangle$. The quotient group $\tilde{E} := E/Z(E) \cong \mathbb{F}_p^{2m}$ is elementary abelian of order p^{2m} and thus a vector space over $\mathbb{Z}_p$ [10, §2, §4 and §11].

Given two elements of E, $e = \mathrm{i}^{\alpha\delta_p}\xi^\lambda X(a)Y(b)$ and $e' = \mathrm{i}^{\alpha'\delta_p}\xi^{\lambda'} X(c)Y(d)$, the following images in $\tilde{E}$: $\tilde{e} = (a|b)$, $\tilde{e'} = (c|d)$ are associated.

Regarding $Z(E)$ as $\mathbb{F}_{p^{1+\delta_p}}$ and $\tilde{E} = E/Z(E)$ as a vector space over $\mathbb{F}_p$, the commutator
$$f : \tilde{E} \times \tilde{E} \ \to\ Z(E) \tag{9}$$
$$(\tilde{e}_1, \tilde{e}_2) \ \mapsto\ [e_1, e_2] \tag{10}$$
leads to the following inner product on $\mathbb{F}_p^{2m}$: $((a|b), (c|d)) = c^{\mathrm{T}}b - a^{\mathrm{T}}d$. Two elements in E commute if and only if their images in $\tilde{E}$ are orthogonal with respect to this inner product. A subspace $\tilde{S}$ of $\tilde{E}$ is said to be *totally isotropic* if for all $\tilde{s_1}$, $\tilde{s_2} \in \tilde{S}$ the symplectic inner product $(\tilde{s_1}, \tilde{s_2}) = 0$. The dimension of a totally isotropic subpace is at most m, and if $\dim \tilde{S} = m$ then $\tilde{S}$ is said to be *maximally totally isotropic*. For example the m-spaces $\tilde{X}$, $\tilde{Y}$ are totally isotropic. It follows that a subgroup $S \subseteq E$ is abelian if and only if its image $\tilde{S}$ in $\tilde{E}$ is totally isotropic.

B. Construction from totally isotropic subspaces

Given a maximally totally isotropic space $\tilde{T}$, the set T of all the preimages is an abelian subgroup of E of order $p^{m+1+\delta_p}$. Thus T is a reducible representation associated with p^m invariant (stable under the action of T) orthogonal lines in $\mathbb{C}^{p^m}$ forming an orthogonal frame $\mathcal{F}(T)$ [10]. If $\tilde{S} \subseteq \tilde{T}$ has dimension n, its preimage $S \subseteq T$ has p^n distinct p^{m-n}-dimensional invariant subspaces of $\mathbb{C}^{p^m}$, each of which is spanned by vectors of $\mathcal{F}(T)$.

The following proposition generalizes [9, Theorem 1].

Proposition 2: Given $k \le m - 1$, the set of all invariant subspaces of the preimages of all $(m-k)$-dimensional totally singular subspaces of $\tilde{E}$ is a packing of N planes in $\mathrm{G}^{\mathbb{C}}_{p^m,p^k}$ with minimal chordal distance $\delta^2 = p^k\left(\frac{p-1}{p}\right)$, where
$$N = p^{m-k} \prod_{j=0}^{m-k-1} \frac{p^{2(m-j)} - 1}{p^{j+1} - 1}.$$

Proof: The proof is an extension of the real case [9] to complex Grassmannians. The number of self-orthogonal codes is provided in [13] and references therein. ∎

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To obtain the packings, projectors associated to invariant spaces can be computed from the projection formula of representation theory [19]. Only the linear characters which satisfy $\chi(\mathrm{i}^{\delta_p}\xi e) = \mathrm{i}^{\delta_p}\xi\chi(e)$ are of interest. Given an abelian subgroup S and a character χ, the associated projector is

$$\Pi = \frac{1}{|S|}\sum_{g\in S}\overline{\chi(g)}g.$$

C. Optimality, EP constraint and finite alphabet generation.

In addition to being well structured, the constructed codebooks possess interesting properties. First, the construction in Prop. 2 leads to several optimum packings.

Corollary 3: For $k = m-1$, the packing of Prop. 2 is an optimal packing of $N = \frac{p(p^{2m}-1)}{p-1}$ planes in $\mathrm{G}^{\mathbb{C}}_{p^m,p^{m-1}}$ with minimal chordal distance $\delta^2 = p^{m-1}\left(\frac{p-1}{p}\right)$ meeting the orthoplex bound (6).

Another interesting aspect is that the entries of the codewords may be selected from a finite alphabet set:

Proposition 3: Representatives of the codewords in Prop. 2 can be chosen such that all their entries are taken from the set $\{0, \langle \mathrm{i}^{\delta_p}\xi\rangle\}$ up to a scaling factor.

Finally, the following proposition gives a restriction of Prop. 2 to derive equal power per-antenna codebooks.

Proposition 4: Given $k \leq m-1$, the set of all invariant subspaces of the preimages S of all $(m-k)$-dimensional totally singular subspaces of $\tilde{S}$ of $\tilde{E}$ such that $\tilde{S}\cap\tilde{Y} = \{0\}$ is a packing of N_{EP} planes in $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{p^m,p^k}$ with minimal chordal distance $\delta^2 = p^k\left(\frac{p-1}{p}\right)$, where

$$N_{EP} = p^{m-k}\prod_{j=0}^{m-k-1}\frac{p^{2(m-j)}-p^{m-j}}{p^{j+1}-1}.$$

Similarly, some of the equal power per antenna codebooks constructed according to Prop. 4 are optimal. We have

Corollary 4: For $k = m-1$, the packing of Theorem 4 is an optimal packing of $N_{EP} = p^{m+1}(p^m-1)/(p-1)$ planes in $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{p^m,p^{m-1}}$ with minimal chordal distance $\delta^2 = p^{m-1}\left(\frac{p-1}{p}\right)$ meeting the orthoplex bound (8).

It is worth noticing that for both Cor. 3 and Cor. 4 with $p = 2$, the bound is reached with the highest possible cardinality (i.e. $N = 2D$ and $N_{EP} = 2D'$, respectively).

V. Applications to MIMO precoding

From the precoding codebook design perspective, when the number of transmit antennas is a power of a prime, Prop. 2 provides codebooks with large distance property and appropriate cardinality. Since the invariant subspaces of higher dimension are spanned by the orthogonal frame of the corresponding maximally isotropic space, a higher-rank codebook can be used for lower-rank transmission. The entries of the codebook can be selected from the finite alphabet set up to a scaling factor. Furthermore, Prop. 4 gives a restriction of Prop. 2 to derive codebooks satisfying the equal power per-antenna constraint. Examples for 2, 3 and 4 transmit antennas are provided. The codebooks are provided in unnormalized matrix form. Orthogonal lines which are invariant subspaces of the same subgroup are presented in a common matrix. Codebooks A correspond to antenna subset selection. Codebooks B and C are subcodebooks related to the equal power per-antenna constraint. Their distance properties or optimality are summarized below.

A. Example for 2 transmit antennas

For two transmit antennas, codebooks with $p = 2$ $m = 1$ and $k = 0$ are relevant. Prop. 2 leads to the codebook consisting of all the columns of {A, B} in Table I. The corresponding normalized codebook is { A, B/$\sqrt{2}$}. According to Corollary 3, this gives an optimum 6-point packing in $\mathrm{G}^{\mathbb{C}}_{2,1}$ meeting the orthoplex bound (6).

Prop. 4 leads to the codebook consisting of all the columns of B. This leads to an optimum 4-point packing in $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{4,2}$ meeting the power-constrained orthoplex bound (8) according to Corollary 4. This codebook is the well-known Mode 1 codebook of WCDMA [20].

A	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
B	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$

TABLE I

Codebooks for 2 transmit antennas

B. Example for 3 transmit antennas

For three transmit antennas, codebooks with $p = 3$ $m = 1$ and $k = 0$ are relevant.

Prop. 2 leads to the codebook consisting of all the columns of {A, B} in Table II. The corresponding normalized codebook is { A, B/$\sqrt{3}$ }. According to Corollary 3, this is an optimum 12-point packing in $\mathrm{G}^{\mathbb{C}}_{3,1}$ meeting the orthoplex bound (6).

Prop. 4 leads to the codebook consisting of all the columns of B. This leads to an optimum 9-point packing in $\mathrm{G}^{\mathbb{C},\mathrm{EP}}_{3,1}$ meeting the power-constrained orthoplex bound (8) according to Corollary 4.

Codebooks for 2-stream transmission with identical cardinality and distance as above can be obtained by taking all the possible combinations of two columns in the same matrix.

A		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
B	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \xi^2 & \xi \\ 1 & \xi & \xi^2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ \xi^2 & \xi & 1 \\ 1 & \xi & \xi^2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ \xi & 1 & \xi^2 \\ 1 & \xi & \xi^2 \end{bmatrix}$

TABLE II

Codebooks for 3 transmit antennas. Here ξ is a third root of unity, e.g. $\xi = \mathrm{e}^{2\pi\mathrm{i}/3}$

C. Example for 4 transmit antennas

For four transmit antennas, codebooks with $m = 2$, $k = 0$ and $k = 1$ correspond to 4 transmit antennas with 1- and 2-stream transmissions respectively.

Prop. 2 with $k = 1$ leads to the 4×2 matrices in {A, B, C} in Table III, and with $k = 0$ to the codebook consisting of all the columns of {A, B, C}. The corresponding normalized codebooks are {1/$\sqrt{2}$ A, B/$\sqrt{2}$, C/2 }. The 4×2 codebook

A	$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$	
	$\left[\begin{array}{cc:cc} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ i & i & 0 & 0 \\ 0 & 0 & -i & i \end{array}\right]$
B	$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$
	$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \\ -i & 0 & i & 0 \end{array}\right]$
C	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -i & -i & i & i \\ -i & i & i & -i \end{array}\right]$
	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ -i & i & -i & i \\ 1 & 1 & -1 & -1 \\ -i & i & i & -i \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ -i & -i & i & i \\ -i & i & -i & i \\ -1 & 1 & 1 & -1 \end{array}\right]$
	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ -i & -i & i & i \\ -i & i & -i & i \\ 1 & -1 & -1 & 1 \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ -i & i & i & -i \\ 1 & -1 & 1 & -1 \\ i & i & -i & -i \end{array}\right]$
	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{array}\right]$	$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -i & -i & i & i \\ i & -i & -i & i \end{array}\right]$

TABLE III
CODEBOOKS FOR 4 TRANSMIT ANTENNAS

in {A, B, C} is an optimum 30-point packing in $G^{\mathbb{C}}_{4,2}$ meeting the orthoplex bound (6) with the maximum possible cardinality according to Corollary 3. Taking the columns of {A, B, C} also gives a 60-point packing in $G^{\mathbb{C}}_{4,1}$ with minimum distance $\delta^2 = 1/2$.

Prop. 4 with $k = 1$ leads to the 4×2 matrices in {B, C}, and with $k = 0$ to the codebook consisting of all the columns of {B, C}. The 4×2 codebook in {B, C} is an optimum 24-point packing in $G^{\mathbb{C},\mathrm{EP}}_{4,2}$ meeting the power-constrained orthoplex bound (8) with the maximum possible cardinality according to Corollary 4. Taking the columns of {B, C} also gives a 48-point packing in $G^{\mathbb{C}}_{4,1}$ with minimum distance $\delta^2 = 1/2$.

The 4×2 codebook in C is an optimum 16-point (4-bit) packing in $G^{\mathbb{C},\mathrm{EP}}_{4,2}$ meeting the power constrained orthoplex bound (8) and 32-point (5-bit) packing in $G^{\mathbb{C},\mathrm{EP}}_{4,1}$ with minimum distance $\delta^2 = 1/2$.

Codebook A corresponds to antenna subset selection. Codebook {B, C} satisfies the equal power per-antenna constraint for 2-stream transmission, while the codebook C satisfies the equal power per-antenna constraint for both 1-stream and 2-stream transmissions. Codebooks for 3-stream transmission with identical cardinality and distance as above are obtained by taking all possible combinations of three columns in the same matrix.

It is worth noticing that the structured codebooks in [7] for 4 transmit antennas are rotated subcodebooks of Table III. For multi-mode codebooks, applying the construction from [9] permits higher rank codebooks that are rotated so that none of the generating lines are overlapping. Corresponding implementation properties and performance can be found in [7].

D. More antennas

For more transmit antennas, the proposed construction yields further codebooks that may be of interest. For example, for selecting 4D subspaces for transmission from 8 antennas, an optimum orthoplex codebook of cardinality 126 is found, with entries in $\{0, \pm 1, \pm \mathrm{i}\}$, and with an equal power restriction of cardinality 112. Restricting the codebook such that equal power per-antenna is satisfied for all possible streams leads to a codebook consisting of 64 unitary matrices: i.e. 1, 2, 4-streams with 9, 8, 7 bits respectively.

VI. CONCLUSION

We have presented a construction of Grassmannian packings related to representation theory which can be applied to build MIMO precoding codebooks. With the chordal distance, the Grassmann manifold can be isometrically embedded in a hypersphere leading to appropriate bounds on the minimum distance. Restricting the codebook design to equal power per-antenna solutions leads to a slight modification of this embedding and the corresponding bounds. We exploit these bounds to prove the optimality of several of the constructed codebooks.

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